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UNITARY DILATIONS AND THE C^* ALGEBRA $\mathcal{O}_2$[†]

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ABSTRACT

Let H be an infinite-dimensional separable Hilbert space, and let $S = (S_{ij}) \in B(H) \otimes M_2$ be a unitary 2×2 matrix with operator entries. We study the C^*-algebra generated by the operators S_{ij}, and show that the study of unitary dilations of isometries T in H reduces to the special case where $S_{11} = T$, and $S_{21} = 0$. We use C^*-algebraic techniques to obtain detailed results about the set of all unitary dilations of T.

§1. Introduction

We study the C^*-algebra generated by a pair of operators S_1, S_2 on an infinite-dimensional separable Hilbert space H, satisfying

$$S_1^* S_1 = P, \tag{1}$$

$$S_2^* S_2 = 1, \tag{2}$$

$$S_i^* S_j = 0, \tag{3}$$

for $i \neq j$, and

$$S_1 S_1^* + S_2 S_2^* = 1 \tag{4}$$

where P is an arbitrary but fixed projection, and 1 denotes the identity operator in H. It will further be assumed that P is the defect projection of a third operator T, which will be assumed isometric but nonunitary. Hence $P = 1 - TT^* > 0$ and $T^* T = 1$.

For the Cuntz algebra $\mathcal{O}_n$, [Cu 1], the axioms are

$$S_i^* S_j = \delta_{ij} 1 \quad \text{and} \quad \Sigma S_i S_i^* = 1.$$

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It follows that the only modification in the case $n = 2$ is the relation $S_1^* S_1 = P < 1$, which replaces $S_1^* S_1 = 1$.

Our work is motivated in part by a desire to understand the symmetries of a class of C^*-algebras which is related to the Cuntz algebras but different. Secondly, we show a direct connection between dilation theory and the C^*-algebra $\mathcal{O}_2$.

The symmetries of $\mathcal{O}_n$ have been studied in recent papers [BEGJ, CE, Vo] where it is shown that the Lie group $U(1, n)$ plays a central role.

The above problem is a special case of the following one:

Let $S = (S_{ij}) \in B(H) \otimes M_2$ be a 2×2 matrix of operators, $S_{ij} \in B(H)$, and assume that S is unitary, i.e., unitary as an operator on $H \oplus H$. What can be said about the C^*-algebra generated by the entries?

For given

$$S = (S_{ij}) \in B(H) \otimes M_{p,q},$$

i.e., a $p \times q$ matrix with $B(H)$-entries, the conditions are

$$S^*S = 1_q \quad \text{and} \quad SS^* = 1_p,$$

where 1_q, resp. 1_p, denotes the identity matrix in q, resp. p, dimensions. Moreover, the case $p = 1$, $q = n$ reduces, as is easily seen, to the relations for Cuntz's $\mathcal{O}_n$. The C^*-algebra is well known to be simple then [Cu 1], but not so in general when $p \neq 1$.

The above-mentioned special case of the 2×2 operator matrices can be understood in terms of dilation theory.

Consider a nonunitary isometry T on H, with $P = 1 - TT^*$, and assume that the matrix

$$S = \begin{pmatrix} T & B \\ A & C \end{pmatrix} \in B(H) \otimes M_2 \tag{5}$$

is unitary. Then it follows that $A = 0$, and the two operators S_1, S_2, defined by $S_1 = B^*$ and $S_2 = C^*$, satisfy the relations (1) thru (4) above. The easy proof is left to the reader.

In the next section, we shall consider unitary matrices of the form (5). We shall write

$$S = \begin{pmatrix} T & S_1^* \\ 0 & S_2^* \end{pmatrix}, \tag{6}$$

and it will be understood, implicitly, then that the entries S_i satisfy relations (1)–(4).

The contents of the paper are as follows: In §2, we consider the order properties on the set of all unitary dilations, $\mathcal{U}_T$, of a given nonunitary isometry T. We show that an element S in $\mathcal{U}_T$ is minimal (in the sense of power dilations) if and only if the operator S_2 (in formula (6) above) is a unilateral shift.

In §3, we construct elements $S = (S_1, S_2)$ in $\mathcal{U}_T$ satisfying $S_1S_1^* \leqq M_1M_1^*$ where T is a given nonunitary isometry in a Hilbert space H, and $M \in \mathcal{U}_T$ is fixed at the outset. We show that the subset of such elements S in $\mathcal{U}_T$ is parametrized by the contraction operators V in H, satisfying $V^*PV = P$, through $S_1 = M_1V$. Moreover, the equality $S_1S_1^* = M_1M_1^*$ holds, if and only if the two operators VV^* and P commute, and $P \leqq VV^*$.

In §4, we consider the C^*-algebra $C^*(S_1, S_2)$ generated by the two operators S_1 and S_2 when it is given that $S = \{S_1, S_2\} \in \mathcal{U}_T$. If Q denotes the range projection of T, i.e., $Q = TT^*$, we consider the ideal I in $C^*(S_1, S_2)$ generated by Q. We prove that the two cases:

(i) $$I = C^*(S_1, S_2)$$

or

(ii) $$\frac{C^*(S_1, S_2)}{I} \simeq \mathcal{O}_2$$

can occur, showing that the quotient is either 0, or else infinite, more specifically, isomorphic to $\mathcal{O}_2$.

Remark 1.1. Let T be a nonunitary isometry (as above) with $P = 1 - TT^*$, and let $\{S_1, S_2\} \in \mathcal{U}_T$. In the very special case where P is equal to one of the two projections $S_1S_1^*$, or $S_2S_2^*$, then the C^*-algebra $C^*(S_1, S_2)$ is of the form $\mathcal{O}_A$. These algebras are studied in [CK] and [Ev 2]. The 2×2 matrix A is $\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$, resp. $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$, if $P = S_1S_1^*$, resp. $P = S_2S_2^*$. Since the second matrix is irreducible, $\mathcal{O}_A$ is simple in this case [CK].

(An example of the first case is $S_1 = P$ and $S_2 = T$. Then $C^*(S_1, S_2) = C^*(T)$ is the nonsimple C^*-algebra studied by Coburn [Co].)

§2. Properties of the operators S_1 and S_2

The conditions (1)–(4) above, on the operator pair S_1, S_2, may be summarized as follows:

(1.1) $$S_i^*S_j = \begin{cases} P & \text{for } i = j = 1 \\ 0 & \text{for } i \neq j \\ 1 & \text{for } i = j = 2 \end{cases}$$

and

$$\sum_{i=1}^{2} S_i S_i^* = 1. \tag{1.2}$$

The projection P is given by $P = 1 - TT^*$ where T is a fixed isometry in the Hilbert space H. When T is given, we shall denote by $\mathcal{U}_T$ the class of all operator pairs $\{S_1, S_2\}$ satisfying the above conditions with $P = 1 - TT^*$. We shall refer to $\{S_1, S_2\}$ as a *dilation pair* for T. Recall that the matrix

$$S = \begin{pmatrix} T & S_1^* \\ 0 & S_2^* \end{pmatrix}$$

represents a unitary operator, S, on $H \oplus H$, and S is a unitary dilation of T. Moreover $H \subset H \oplus H$ is an invariant subspace for S, but not for S^*. An easy calculation proves that S is a power dilation of T (cf. [SF]). In general, however, S may be different from the minimal power dilation of T.

We have the following:

THEOREM 2.1. *The operator*

$$\begin{pmatrix} T & S_1^* \\ 0 & S_2^* \end{pmatrix}$$

on $H \oplus H$ is the minimal *unitary power dilation of the given nonunitary isometry T on H if and only if S_2 is a unilateral shift.* (*It is assumed that S_1 and S_2 satisfy* (1.1)–(1.2) *above.*)

The proof depends on the following:

LEMMA 2.2. *Let V be an isometry in a Hilbert space H, and let $\overline{R(V)}$ denote the closure of the range of V, i.e., the range of the projection VV^*. Then V is a unilateral shift if and only if the only vector z in H which satisfies*:

$$V^{*n} z \in \overline{R(V)} \qquad \textit{for } n = 0, 1, \ldots \tag{$*$}$$

is $z = 0$.

PROOF. Suppose V is a unilateral shift [Ha], and that $L \subset H$ is a wandering subspace which is generating, cf. [SF, p. 2], and [Ha]. We have

$$H = \sum_0^\infty \oplus V^n L.$$

Assume that $z \in H$ decomposes:

$$z = (z_0, z_1, \ldots), \qquad z_n \in V^nL,$$

and satisfies condition (*) in Lemma 2.2. It follows that

$$V^*z = (z_1, z_2, \ldots),$$

$$V^{*2}z = (z_2, z_3, \ldots),$$

etc. We conclude that $z_0 = 0$ since $z \in \overline{R(V)}$, $z_1 = 0$ since $V^*z \in \overline{R(V)}$, and, by induction, $z_n = 0$ for all $n \geqq 0$.

This proves that $z = 0$.

Assume, conversely, that only the vector $z = 0$ satisfies condition (*). Consider the Wold decomposition [SF, Theorem 1.1] for the isometry V: There is a decomposition $H = H_1 \oplus H_2$ which reduces V such that $A = V|_{H_1}$ is a unilateral shift, and $U = V|_{H_2}$ is unitary on H_2.

Let $z = (z_1, z_2)$ denote the components of z relative to the Wold decomposition. We have: $V^{*n}z = (A^{*n}z_1, U^{-n}z_2)$ for $n = 0, 1, \ldots$. Moreover, $\overline{R(V)} = \overline{R(A)} \oplus H_2$. It follows that condition (*) is satisfied for all vectors of the form $(0, z_2)$. In particular, (*) holds for nonzero vectors only if $H_2 \neq 0$, i.e., V is not a unilateral shift.

Proof of Theorem 2.1. In view of Lemma 2.2, it is enough to apply the condition in the lemma to the isometry S_2. Let

$$S = \begin{pmatrix} T & S_1^* \\ 0 & S_2^* \end{pmatrix},$$

and assume that $y \oplus z \in H \oplus H$ is orthogonal to $S^{*n}(H \oplus (0))$ for all $n \geqq 0$. Then, of course, $y = 0$, and an induction shows that $S_1^* S_2^{*n} z = 0$ for all $n = 0, 1, \ldots$. Since the null space for S_1^* is precisely $\overline{R(S_2)}$, it follows that $S_2^{*n}z \in \overline{R(S_2)}$ for $n \geqq 0$, if and only if $0 \oplus z$ is orthogonal to the spaces $S^{*n}(H \oplus (0))$ for all $n \geqq 0$. Hence, the dilation S is a minimal power dilation of T precisely when the isometry S_2 is a unilateral shift.

Corollary 2.3. *Let T be a unilateral shift in a Hilbert space H, and let $P = 1 - TT^*$. Then $\mathcal{U}_T$ is parametrized by the group of all unitary operators V in H as follows: $S_1 = VP$ and $S_2 = VTV^*$.*

Proof. Since T is a unilateral shift, $(P, T) \in \mathcal{U}_T$ satisfies the condition in Theorem 2.1. Hence $\left(\begin{smallmatrix} T & P \\ 0 & T^* \end{smallmatrix}\right)$ is a minimal unitary power dilation of T. Since such dilations are unique, up to unitary equivalence ([SF, Theorem 4.2]), the corollary follows.

For a given isometry T, we shall now discuss the set of all unitary dilations, $\mathcal{U}_T$. For an element $\{S_1, S_2\}$ in $\mathcal{U}_T$, the first operator, S_1, is a partial isometry with initial space P and final space $S_1S_1^*$. (We shall adopt the convention of identifying projections, $Q = Q^*$ on H with the spaces $QH = \{x \in H\colon Qx = x\}$.)

Definition 2.4. For two elements $S = \{S_1, S_2\}$ and $S' = \{S_1', S_2'\}$, in $\mathcal{U}_T$, we shall say that $S \leqq S'$ if the graph of S_1^* on its initial space is contained in that of $S_1'^*$. Let the final projections of S_1, resp. S_1', be denoted Q, resp. Q', i.e., $S_1S_1^* = Q$, resp. $S_1'S_1'^* = Q'$. Then it follows that $S \leqq S'$ if and only if

$$Q = Q'Q \tag{2.1}$$

and

$$S_1'^*Q = S_1^*. \tag{2.2}$$

The following remark, which is immediate from 2.4, shows that the dilation pairs $\{S_1, S_2\}$ display a certain rigidity. It also shows that $\leqq$ is in fact an equivalence relation.

Remark 2.5. Let T be a non-unitary contraction, and let S and S' be two elements in $\mathcal{U}_T$. If $S \leqq S'$, it follows that $S_1 = S_1'$. In this case, we say that S and S' are equivalent. The two isometries S_2 and S_2' may be different, but they have common final projection, viz. $1 - S_1S_1^*$.

Proof. Assume $S \leqq S'$. Then $S_1'^*$ is one-to-one on its initial space. Its final space is P which is also the final space of S_1^*. Thus, if $S \leqq S'$, we must have $S_1^* = S_1'^*$.

We now turn to a particular element in $\mathcal{U}_T$ which is canonically given in terms of T.

The element $\{S_1, S_2\}$, given by $S_1 = P$, and $S_2 = T$, is called the *canonical dilation pair*, corresponding to the unitary dilation $\begin{pmatrix} T & P \\ 0 & T^* \end{pmatrix}$ for a given isometry T. Recall, $P = 1 - TT^*$.

We shall need the following:

Proposition 2.6. *Let T be a nonunitary isometry in H with infinite defect projection P. For an arbitrary element $S = (S_1, S_2) \in \mathcal{U}_T$, the following are equivalent:*

$$P \leqq S_1S_1^*. \tag{3}$$

$$T^*S_2 \text{ is an isometry.} \tag{4}$$

There is an isometry V on H such that

(5) $$S_2 = TV.$$

Moreover, every isometry V gives rise to a dilation pair, $S = (S_1, TV)$, satisfying (3).

REMARK 2.7. Note that the first part of the proposition holds without the restriction $\dim P = \infty$.

Before starting the proof we recall the following known lemma [Fu, Lemma 6] which will be used in the next two results.

LEMMA 2.8. *Let E, F, and G be self-adjoint projections in a Hilbert space H, and assume*

$$EFE = G.$$

Then it follows that the three projections mutually commute, and

$$G = E \wedge F.$$

PROOF [Fu]. The operator $[E, F] = EF - FE$ is a skew-adjoint and satisfies $[E, F]^3 = 0$.

PROOF OF PROPOSITION 2.6. Let $S = \{S_1, S_2\} \in \mathcal{U}_T$ be given, and assume (3). Then $S_2S_2^* = 1 - S_1S_1^* \leqq 1 - P = TT^*$, and it follows that there is a contraction operator V on H satisfying $S_2^* = V^*T^*$. Substitution of this into the matrix formula

$$\begin{pmatrix} T & S_1^* \\ 0 & S_2^* \end{pmatrix} \begin{pmatrix} T^* & 0 \\ S_1 & S_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

yields $V^*T^*TV = 1$. Since $T^*T = 1$, it follows that $V^*V = 1$, proving (5).

Assuming $S_2 = TV$ for an isometry V, we get, upon multiplying by T^* on the left, $T^*S_2 = T^*TV = V$, proving that T^*S_2 is isometric.

If T^*S_2 is assumed to be an isometry, then

$$(T^*S_2)^*T^*S_2 = 1$$

which implies

$$(S_2S_2^*)(TT^*)(S_2S_2^*) = S_2S_2^*.$$

It follows (Lemma 2.8) that the projections $S_2S_2^*$ and TT^* commute, and $S_2S_2^* \leqq TT^*$ which is equivalent to (3).

Conversely, let V be an isometry such that (5) holds. Then, $S_2S_2^* = TVV^*T^* \leqq TT^* = 1 - P$, and therefore, $1 - S_2S_2^* \geqq P$. Since P is assumed infinite-dimensional there is a partial isometry S_1 with initial projection P and final projection $1 - S_2S_2^*$. It follows that $\{S_1, TV\} \in \mathcal{U}_T$, and this pair satisfies (3).

The first part of Proposition 2.6 above has a dual:

PROPOSITION 2.9. *Let T be a nonunitary isometry in H with deficiency projection P. For an arbitrary $S = \{S_1, S_2\} \in \mathcal{U}_T$, the following are equivalent:*

(3′) $$S_1S_1^* \leqq P.$$

(4′) $$S_2^*T \text{ is an isometry.}$$

(5′) *There is an isometry W such that*

$$T = S_2W.$$

PROOF. Note that $\{S_1, S_2\} \in \mathcal{U}_T$ implies $\{S_1^*, T\} \in \mathcal{U}_{S_2}$. Thus, Proposition 2.9 follows readily from the first part of Proposition 2.6 (and Remark 2.7).

REMARK 2.10. Let T be an isometry, and assume that $P = 1 - TT^*$ is infinite-dimensional. Let V be a second isometry.

Then there is a dilation pair $\{S_1, S_2\} \in \mathcal{U}_T$ such that $T = S_2V$, i.e., the conditions in Proposition 2.9 are satisfied.

PROOF. Both of the final projections, VV^* and TT^*, are infinite-dimensional. Hence, there is an isometric transformation of one *onto* the other. We shall extend this transformation, with domain $= (VV^*)H$, to become an isometry defined on all of H and satisfying $S_2V = T$. When extended, it will be denoted S_2. Since the complement of TT^*, viz. P, is infinite-dimensional, S_2 can be constructed in such a way that $1 - S_2S_2^*$ is still infinite-dimensional. Now let S_1 be a partial isometry with P as initial space, and with $1 - S_2S_2^*$ as final space. It is clear that the pair $\{S_1, S_2\}$ lies in $\mathcal{U}_T$ and that the conditions, $S_2V = T$ and $S_1S_1^* \leqq P$, are both satisfied.

COROLLARY 2.11. *Let T be a given nonunitary isometry with defect projection P, and let $\{S_1, S_2\} \in \mathcal{U}_T$ be given. Then the following are equivalent:*

(3″) $$P = S_1S_1^*$$

and

(4″) $$S_2^*T \text{ is unitary.}$$

PROOF. The corollary follows when Propositions 2.6 and 2.9 are combined. If (3″) holds then both of the operators T^*S_2 and $S_2^*T = (T^*S_2)^*$ are isometric which means that T^*S_2 is unitary. The implications may be reversed, and the two propositions apply again to give the converse, (4″) $\Rightarrow$ (3″).

COROLLARY 2.12. *Let T be a nonunitary isometry on H with deficiency projection P. Let $\{S_1, S_2\} \in \mathscr{U}_T$ be such that*

$$\{P, T\} \leqq \{S_1, S_2\}.$$

Then $S_1 = P$, and there is a unitary operator U on H such that

$$S_2 = TU.$$

Conversely, if U is any unitary, then $\{P, TU\} \in \mathscr{U}_T$, and

$$\{P, T\} \leqq \{P, TU\}$$

holds.

PROOF. The proof is immediate form the propositions, and it is left to the reader.

§3. Construction of different pairs S_1, S_2

The standing assumption is that H is an infinite-dimensional separable Hilbert space, and T is a nonunitary isometry in H with $P = 1 - TT^*$. If, in addition, S_1 is a partial isometry with $S_1^*S_1 = P$, then, clearly, an isometry S_2 exists, such that $\{S_1, S_2\} \in \mathscr{U}_T$, if and only if $1 - S_1S_1^*$ is infinite-dimensional.

If, conversely, S_2 is a given isometry, then a partial isometry S_2 exists such that $\{S_2, S_2\} \in \mathscr{U}_T$ if and only if the two projections $1 - S_2S_2^*$ and P have the same dimension.

It follows that, if $\{S_1, S_2\} \in \mathscr{U}_T$, and if $\dim P < \infty$, then there is no $\{S_1', S_2'\} \in \mathscr{U}_T$ such that

$$S_1'S_1'^* < S_1S_1^*$$

or

$$S_1'S_1'^* > S_1S_1^*.$$

We will show in this section that the situation in less rigid if $\dim P = \infty$ and we shall classify the possibilities.

When comparing two elements in $\mathscr{U}_T$, we restrict attention to only the first of the two conditions, (2.1) and (2.2), in Definition 2.4.

THEOREM 3.1. *Let T be a non-unitary isometry in H with $P = 1 - TT^*$, and let $M = \{M_1, M_2\} \in \mathcal{U}_T$ be given.*

(i) *Let V be a contraction operator, i.e., $V \in B(H)$, $\|V\| \leqq 1$, satisfying $V^*PV = P$. Then there exists an isometry S_2 such that the pair $S = \{M_1V, S_2\}$ is in $\mathcal{U}_T$, and $S_1S_1^* \leqq M_1M_1^*$.*

(ii) *Conversely, for every $S = \{S_1, S_2\} \in \mathcal{U}_T$ satisfying*

$$S_1S_1^* \leqq M_1M_1^*, \tag{$*$}$$

there is a contraction V such that $S_1 = M_1V$ and $V^*PV = P$.

(iii) *Equality holds in $(*)$, i.e., $S_1S_1^* = M_1M_1^*$, if and only if the operators P and VV^* commute, and $P \leqq VV^*$; in fact, $P(VV^*) = P$.*

PROOF. (i) Let $M = \{M_1, M_2\} \in \mathcal{U}_T$, and let V satisfy the assumptions. Define $S_1 = M_1V$. Then

$$S_1^*S_1 = V^*M_1^*M_1V = V^*PV = P.$$

It follows that S_1 is a partial isometry. In particular, $S_1S_1^*$ is a projection, viz. the final projection of S_1. We have $S_1S_1^* = M_1VV^*M_1^* \leqq M_1M_1^*$ since $VV^* \leqq 1$. Hence, $1 - S_1S_1^* \geqq 1 - M_1M_1^* = M_2M_2^*$. It follows that $1 - S_1S_1^*$ is infinite-dimensional and, therefore, there is an isometry S_2 with $1 - S_1S_1^*$ as its final projection. We have proved that this pair $\{S_1, S_2\}$ lies in $\mathcal{U}_T$, and satisfies the inequality $(*)$.

(ii) If conversely $\{S_1, S_2\} \in \mathcal{U}_T$, and $(*)$ holds, then a contraction operator V exists such that $S_1^* = V^*M_1^*$. We also have $P = S_1^*S_1 = V^*M_1^*M_1V = V^*PV$. This proves part (ii) of the theorem.

(iii) Assume equality in $(*)$ and substitute $S_1 = M_1V$. We get

$$M_1M_1^* = S_1S_1^* = M_1VV^*M_1^*.$$

Now multiply through with M_1^* on the left and M_1 on the right in this formula. We get

$$M_1^*M_1 = (M_1^*M_1)(VV^*)(M_1^*M_1).$$

Since $P = M_1^*M_1$ this amounts to

$$P = P(VV^*)P.$$

For the operator VV^*, we have

$$0 \leqq VV^* \leqq 1. \tag{$\dagger$}$$

Decompose the Hilbert space H relative to the projection P and its complement $1-P$. The matrix for VV^* then takes the form

$$\begin{pmatrix} P & L \\ L^* & Q \end{pmatrix}$$

(relative to the decomposition). For vectors $z \in H$, we write $x = Pz$ and $y = (1-P)z$.

Substitution of this into the estimates listed in (†) above leads to:

$$0 \leqq \|Px\|^2 + 2\operatorname{Re}(x, Ly) + (Qy, y) \leqq \|x\|^2 + \|y\|^2.$$

Since $Px = x$, this reduces in turn to:

$$-\|x\|^2 \leqq 2\operatorname{Re}(x, Ly) + (Qy, y) \leqq \|y\|^2.$$

For arbitrary vectors x, y as above, consider the two complex lines tx in PH ($t \in \mathbf{C}$), and then ty in the complement. The resulting elementary quadratic form estimates yield

$$(x, Ly) = 0 \quad \text{and} \quad 0 \leqq (Qy, y) \leqq \|y\|^2.$$

In particular, $L = 0$ when L is regarded as an operator from $(1-P)H$ to PH. The matrix for VV^* reduces then to $\left(\begin{smallmatrix} P & 0 \\ 0 & Q \end{smallmatrix}\right)$, and the conclusions listed in (iii) become obvious.

The converse implication in (iii) is easy and is left to the reader.

§4. A dichotomy for the C^*-algebra $C^*(S_1, S_2)$

Let T be a nonunitary isometry in a separable infinite-dimensional Hilbert space H, and let $S = \{S_1, S_2\} \in \mathcal{U}_T$, i.e., the matrix

$$\begin{pmatrix} T & S_1^* \\ 0 & S_2^* \end{pmatrix}$$

is a unitary dilation of T. In this final section, we study the unital C^*-algebra $\mathfrak{A}$ generated by the two partial isometries, S_1 and S_2 in H, i.e., $\mathfrak{A} = C^*(S_1, S_2)$.

Let $P = 1 - TT^*$. Since $S_1^* S_1 = P$, it follows that $P \in \mathfrak{A}$, and $P^\perp = 1 - P = TT^* \in \mathfrak{A}$. We consider the closed (two-sided) $*$-ideal in $\mathfrak{A}$ generated by $P^\perp$. The ideal will be denoted I.

THEOREM 4.1. *Let T be a nounitary isometry with defect projection P. If the closed ideal I generated by $P^\perp$ is proper, i.e., $I \neq \mathfrak{A}$, then $\mathfrak{A}/I$ is isomorphic to the Cuntz algebra $\mathcal{O}_2$.*

REMARK 4.2. It follows that there are only two possibilities for $\mathfrak{A}/I$: Either $\mathfrak{A}/I$ is trivial $=0$, or else it is infinite, $\mathfrak{A}/I \simeq \mathcal{O}_2$. Recall [Cu 1] that $\mathcal{O}_2$ is an infinite, simple C^*-algebra.

Before turning to the proof, we list some cases where $\mathfrak{A}/I$ is trivial, i.e., $I = \mathfrak{A}$. The following implications hold: (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Leftrightarrow$ (4), and (5) $\Rightarrow$ (3), where:

$$S_1S_1^* \leqq P \quad \text{and} \quad S_2^*T \in \mathfrak{A}, \tag{1}$$

$$T \in \mathfrak{A}, \tag{2}$$

$$I = \mathfrak{A}, \tag{3}$$

$$S_1 \in I \quad \text{or} \quad S_2 \in I, \tag{4}$$

$$S_1S_1^* \geqq P. \tag{5}$$

PROOF OF THEOREM 4.1. Assume $I \neq \mathfrak{A}$, and let $\varphi: \mathfrak{A} \to \mathfrak{A}/I$ be the quotient-mapping. Let $U_i = \varphi(S_i)$, $i = 1,2$. We have

$$S_i^*S_j = \begin{cases} P & i = j = 1 \\ 0 & i \neq j \\ 1 & i = j = 2 \end{cases}$$

and

$$\sum S_iS_i^* = 1.$$

When φ is applied, the new relations

$$U_i^*U_j = \delta_{ij}1 \quad \text{and} \quad \sum U_iU_i^* = 1$$

follow, where 1 is the identity in $\mathfrak{A}/I$. It follows that both of the elements, U_1 and U_2, are nonzero in $\mathfrak{A}/I$, which means that neither S_1 nor S_2 lies in I.

Now we let $\mathcal{O}_2$ denote a "copy" of the Cuntz algebra with generators σ_i, $i = 1,2$, satisfying $\sigma_i^*\sigma_j = \delta_{ij}1$ and $\Sigma\, \sigma_i\sigma_i^* = 1$. Recall [Cu 2] that $\mathcal{O}_2$ is determined, as a unital C^*-algebra, up to isomorphism. It is simple and infinite.

Using the universal property of $\mathcal{O}_2$, we get a homomorphism $\psi: \mathcal{O}_2 \to \mathfrak{A}/I$, satisfying $\psi(\sigma_i) = U_i \neq 0$, $i = 1,2$. Since the kernel, $\ker\psi$, is a closed ideal, $\ker\psi \neq \mathcal{O}_2$, it follows that ψ is an isomorphism of $\mathcal{O}_2$ onto its image. On the other hand, the two generators U_1 and U_2 lie in the range of ψ, and it follows that $\psi(\mathcal{O}_2) = \mathfrak{A}/I$. This concludes the proof of Theorem 4.1.

PROOF OF REMARK 4.2. Assume the two conditions listed under (1). Let $V = S_2^* T \in \mathfrak{A}$. It follows from Proposition 2.9 that V is an isometry satisfying $T = S_2 V$. Both S_2 and V are in $\mathfrak{A}$, and therefore $T \in \mathfrak{A}$, which proves (2).

Now, assume (2). Since $TT^* = P^\perp$ is the range projection of T, it follows that

$$T^* P^\perp T = T^* T = 1.$$

Since $T \in \mathfrak{A}$, this means that 1 is in the ideal I generated by $P^\perp$. Hence, $I = \mathfrak{A}$.

Clearly (3) $\Rightarrow$ (4), and it follows from the proof of Theorem 4.1 that the converse implication is also valid. That is, if $I \neq \mathfrak{A}$, then $S_1 \notin I$ and $S_2 \notin I$.

We finally assume condition (5), i.e., $S_1 S_1^* \geqq P = 1 - TT^*$. It follows that

$$S_2^* P^\perp S_2 = 1.$$

This is because $P^\perp$ contains the final projection of S_2 when (5) is assumed. Hence, $1 \in I$ and $I = \mathfrak{A}$, which proves (3), and concludes the proof of Remark 4.2.

REMARK 4.3. Recently, other results on $\mathcal{O}_2$ have appeared. They are [Bl, LTW], and came to our attention after the completion of this paper. In [LTW], a pure state extension to $\mathcal{O}_2$ is constructed of a certain non-hyperfinite II_1 factor state on the Choi subalgebra. In [B1], it is proved that every non-type-I C^*-algebra contains a subalgebra which has $\mathcal{O}_2$ as a quotient. It would be interesting to study connections to dilation theory.

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